Problem set 1: Detection tasks and Fourier transforms

Theoretical image science
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Discussing solutions is encouraged but you must **individually** write and hand in your solutions. For numerical computations and plots, you can use the programming language of your choice, e.g. Matlab or Python.

These homework problems build on material covered in **lectures 1 and 2**.

1 SNR and AUC (2p)

Consider a binary classification task and assume that t is a test statistic that is normally distributed under each hypothesis with the same variance σ^2 but different means $\bar{t}_1$ and $\bar{t}_2 > \bar{t}_1$. Prove Eq. 9.10 in [BKVM]:

$$\mathrm{AUC} = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\mathrm{SNR}_t}{2}\right) \tag{1}$$

with

$$\mathrm{SNR}_t = \frac{\bar{t}_2 - \bar{t}_1}{\sigma}. \tag{2}$$

Hint: First let $h_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}$ denote a Gaussian kernel and show that $[H * h_\sigma](x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\frac{x}{\sqrt{2}\sigma}$ where $H(x)$ is the Heaviside step function. Then express 9.9 in the book as a convolution integral and use known properties of convolutions.

2 The ideal observer (2p)

The figure shows the probability density function for a scalar-valued measurement g (a "single-pixel image") under the two truth states (hypotheses) T_1 and T_2.

(a) Derive an expression for the likelihood ratio as a function of g and plot the ROC curve. Would the ROC curve be different with a a test statistic that is linear in g?
(b) Calculate the AUC (you can do this numerically if you want.)

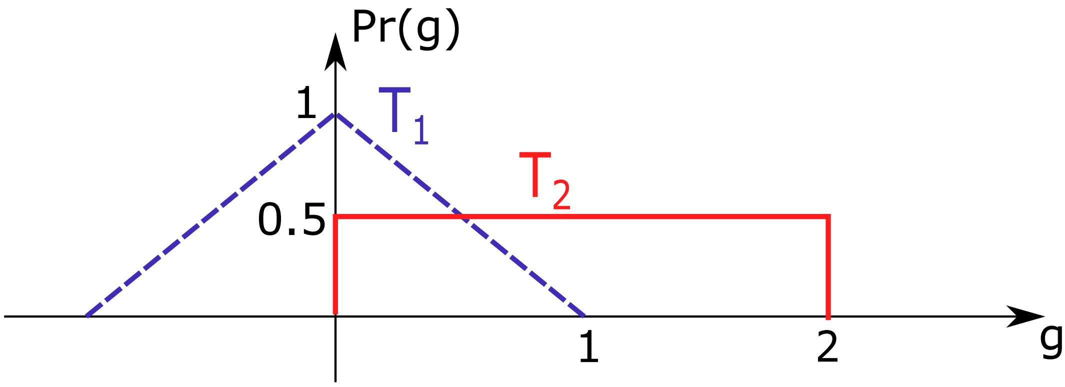

Figure 1 – Probability distribution functions for g in problem 2 under two truth states: T_1 (triangular distribution) and T_2 (rectangular distribution).

(c) Calculate the SNR and the corresponding AUC that would have resulted under the assumption of Gaussian statistics (i.e. the AUC you get by replacing each probability distribution by a Gaussian distribution with the same mean and variance as the given distributions). This can be seen as an approximation of the true AUC. You may look up and use expressions for the variance of the uniform and triangular distributions without deriving them.
(d) In most situations, a nonzero TPF is not possible to get without also getting some false positives. Explain why this is possible in this particular case.

3 The ideal observer in nuclear imaging (2p)

A gamma camera is a detector with a collimator in front that can be used to image the distribution of a radioactive substance in the human body. Consider a gamma camera with N pixels that is used to acquire a 2D image. Pixel i measures $g_i \in \text{Po}(\lambda)$ photons, independently of the other pixels, where lambda is expected photons measured during the scan (same for all pixels). You want to distinguish between truth states $T_1 : \lambda = \lambda_1$ (disease absent) and $T_2 : \lambda = \lambda_2$ (disease present). Assume that $\lambda_2 > \lambda_1$.
(a) Show that the test statistic $\sum_{i=1}^{N} g_i$ is equivalent to the ideal observer.
(b) Now approximate g_i as a Gaussian random variable with the mean λ and variance λ. Derive a formula for the ideal-observer test statistic in this case.
(c) Are there conditions under which the decision strategies in (a) and (b) give a large difference in the result?
Note 1: What we have modeled here is a limiting case where the feature we are looking for is larger than the camera. Note 2: It is very common to approximate probability distributions with Gaussians but in some cases, like here, this approximation can lead to incorrect conclusions.

4 A nonlinear imaging system (2p)

A lot of imaging systems have approximately linear imaging operators $\mathcal{H}$, but there are also situations where $\mathcal{H}$ is nonlinear. An example is systems based on counting particles (e.g. photons, electrons, or protons...). If the count rate is too high, these may start missing counts ("pileup"), leading to a nonlinear relation between input and output counts. In one simple model (a "nonparalyzable" detector) the number of input counts in a given time interval of length t is Poisson distributed with expected value nt and the number of registered counts Y in the same interval has expected value $\langle Y \rangle = nt/(1+n\tau)$ and variance $V[Y] = nt/(1+n\tau)^3$. Here, n is the input count rate (counts/time) and where τ is the time it takes to register one count. (You do not have to prove this!).

(a) Derive a formula for the SNR as a function of n for the task of discriminating between the two truth states T_1: $n = n_1$ and $T_2 : n = n_2$ $(> n_1)$.

(b) From the answer in (a) derive an approximate expression for the SNR in terms of $\Delta n = n_2 - n_1$ that is valid to first order in Δn is when Δn is small relative to τ^{-1}.

(c) Now we want to study the imaging performance under high count rates. Describe the difference between the asymptotic behavior of SNR for the two cases (I) n_1 is fixed while $n_2 \to \infty$, and (II) $\Delta n/n_1$ = constant and small while $n_1 \to \infty$. Which of these SNR figures would you focus on when optimizing the performance of an imaging system? Does your answer vary depending on the application?

5 The Fast Fourier transform (2p)

Assume that you have an image defined on an $N \times N$ grid with spacing x_0 and y_0 covering the interval $[-X, X - x_0] \times [-Y, Y - y_0]$ (X and Y are even multiples of x_0 and y_0).
(a) Write a script that uses the two-dimensional fourier transform to approximate the Fourier transform discretized on an $N \times N$ grid with spacing u_0, v_0. (Hint: use the `fft2`, `fftshift` and `ifftshift` functions available in `Matlab` or `scipy.fft`). Figure out what the frequencies (u, v) are for the coordinate points of the matrix you get from `fft2` in units of x_0^{-1} and y_0^{-1}. Test this function for $N = 100$ by transforming a two-dimensional Gaussian with $\sigma_x = 0.6x_0$ and $\sigma_y = 3y_0$ and verify the result by plotting it together with the analytically calculated transform in plots along the u and v axes .
(b) Express the discrete Fourier transform of the sampled version of a continuous function using the comb function $\text{III}_{x_0,y_0}(x, y) = \sum_{i,j=-\infty}^{\infty} \delta(x - ix_0, y - jy_0)$. Then use this expression to derive an exact mathematical formula for the numerically calculated approximate transform that you plotted in a.

Note: It's easy to make a mistake with the normalization or coordinate axes when Fourier transforming numerically, so you may find this function useful in the future.