Problem set 2: Random processes and noise

Theoretical image science
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Discussing solutions is encouraged but you must **individually** write and hand in your solutions. For numerical computations and plots, you can use the programming language of your choice, e.g. Matlab or Python.

These homework problems build on material covered in **lectures 1-4**.

1 The noise remover (3p)

You have just bought a new telescope that has a built-in camera and a piece of software called a NoiseRemover™. (Sounds highly useful, doesn't it?) When you study the implementation of the NoiseRemover™, you find that it acts on each pixel value g_i of the image $\mathbf{g}$ individually, transforming it to the output pixel value

$$g_i^{\text{out}} = \begin{cases} 1 & \text{if } g_i \geq 0.5 \\ 0 & \text{if } g_i < 0.5 \end{cases} \tag{1}$$

Now you will study its performance for the task of detecting a star in an image of the night sky. If there is a star in pixel i, that pixel has expected intensity $\overline{g}_i = 1$, and otherwise, $\overline{g}_i = 0$. Assume that the background is white (=uncorrelated) Gaussian Noise with standard deviation σ. You may use known properties of the truncated normal distribution (see `https://en.wikipedia.org/wiki/Truncated_normal_distribution`) without deriving them.

(a) Plot the ratio of the SNR of the Hotelling observer ("Hotelling detectability") before and after applying the NoiseRemover™for $\sigma \in [0.1, 10]$. Hint: To simplify expressions, it can be helpful to introduce the notation p for the expected true positive fraction.

(b) The ideal-observer detectability is $d_A = 2\text{erf}^{-1}(2\text{AUC} - 1)$ where AUC is the area under the ROC curve of the ideal observer. Plot the ratio of the ideal-observer detectability before and after applying the NoiseRemover™for $\sigma \in [0.1, 10]$. Comment on the different behavior of the plots in (a) and (b).

2 Imaging beyond the shot-noise limit (2p)

When imaging with light photons or subatomic particles, the signal-to-noise ratio is in most cases limited by the fact that these quanta are emitted at random and independent time points so that the number of emitted quanta N during a given time period is Poisson distributed, $N \in \mathrm{Po}(\lambda)$ where λ is the expected number of emitted quanta. This is called the "quantum" or "shot-noise" limit. However, image quality can be improved beyond this limit if the quanta exhibit subpoisson statistics ("antibunching"). This can be achieved for example in optical fluorescence imaging.

Assume that λ fluorescence photons are emitted during a given time interval, with a variance $F\lambda$ where $0 < F \leq 1$ is the called the Fano factor. You want to discriminate between two cases for which λ differs by 10%.

a) Derive an expression for the $\mathrm{SNR}_{\mathrm{ideal}}$ that you will measure with an ideal detector, as a function of F. What happens if the fluorescence emission is Poisson distributed?
b) Assume that you are using a detector that detects a fluorescence photon with probability η. This means that the number of detected photons N^{det}, given N emitted photons is a Binomial distributed random variable. Derive an expression for $\mathrm{SNR}^2/\mathrm{SNR}^2_{\mathrm{ideal}}$. Also comment on what happens in the Poisson-distributed case. Is a non-unity detection efficiency more detrimental for some values of F?
Depending on what strategy you choose, you may need the law of total variance $V[Y] = \langle V[Y|X]\rangle + V[\langle Y|X\rangle]$.
Note: As we'll see later in the course, $\mathrm{SNR}^2/\mathrm{SNR}^2_{\mathrm{ideal}}$ *is known as the detective quantum efficiency.*

3 Random processes (2p)

A wide-sense stationary random process $a(x, y)$ has mean $\mu_a = 1$ and autocovariance function $K(\Delta x, \Delta y) = \exp(-2|\Delta x| - 3|\Delta y|)$.
(a) Calculate the correlation $\langle a(0, 0) \cdot a(1, 1)\rangle$.
(b) Calculate the NPS.
(c) Calculate the variance of $a(x, y)$ both from $K(\Delta x, \Delta y)$ and from the NPS and show that these give the same answer.

4 Cyclostationarity (1p)

Read in the book about wide-sense cyclostationary (WSCS) random processes. Assume that $a_n, n \in \mathbb{Z}$ is a wide-sense stationary random process in one discrete coordinate with mean μ^a and autocorrelation function $R^a_n, n \in \mathbb{Z}$. Form the continuous-coordinate random process $a(x) = \sum_{n=-\infty}^{\infty} a_n s(x - nx_0)$ where $s(x)$ is called the sensing function.

(a) Show that the mean $\langle a(x) \rangle$ is invariant under translations of a multiple of x_0.
(b) Show that the autocorrelation of $a(x)$ is

$$R_a(x', x' + x) = \langle a(x')a^*(x' + x) \rangle = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} R^a_{n_2-n_1} s(x' - n_1x_0)s^*(x' + x - n_2x_0) \quad (2)$$

Explain why you can conclude that $a(x)$ is WSCS.
Note: this type of random process can be used to model a continuous signal recovered from discrete measurements. In signal processing this is known as pulse amplitude modulation (PAM).

5 Measuring the noise power spectrum (2p)

In this numerical exercise you will learn how to generate correlated noise and characterize it in terms of the noise power spectrum. As an example we will study ramp-filter noise

$$\mathrm{NPS(u, v)} = C\sqrt{u^2 + v^2} \quad (3)$$

where C is a constant. This type of noise commonly occurs in tomographic images.

a) Start by generating an image of 100×100 pixels containing white Gaussian noise with mean 0 and standard deviation 1.
b) Fourier transform this image and multiply with a suitably chosen filter function such that you should get ramp filter noise, still with mean 0 and standard deviation 1, after inverse Fourier transformation. You should also present the derivation of this filter function and provide an expression for the constant C (this may be formulated as an integral).
c) Compare the visual appearance of these noise fields. Can you see a visual difference?
d) Now estimate the NPS from 50 image realizations generated in the way described above. Plot $\mathrm{NPS}(u, 0)$ and compare to the analytical formula (3).